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# VARIATIONAL PRINCIPLES FOR PROBLEMS WITH LINEAR CONSTRAINTS. PRESCRIBED JUMPS AND CONTINUATION TYPE RESTRICTIONS

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# VARIATIONAL PRINCIPLES FOR PROBLEMS WITH LINEAR CONSTRAINTS. PRESCRIBED JUMPS AND CONTINUATION TYPE RESTRICTIONS

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## ABSTRACT

A general theory of problems subjected to linear constraints is developed. As applications, a problem for which the solutions are required to satisfy prescribed jumps and another one whose solutions are restricted to be such that they can be continued smoothly into solutions of given equations in neighboring regions, are formulated abstractly. General variational principles for these types of problem are reported. In addition it is shown that sets of functions that can be extended in the manner explained above constitute, generally, completely regular subspaces, here defined. These results have a bearing on boundary methods which are being developed for treating numerically partial differential equations associated with many problems of Science and Engineering.

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## SIGNIFICANCE AND EXPLANATION

Many boundary value problems involve situations in which either the solutions are refined to satisfy prescribed jump conditions at interior boundaries, or the solutions must be such that they can be continued smoothly into solutions of given equations in a neighboring region. This paper gives an abstract formulation and general variational principles for such problems.

The variational principles are useful in numerical applications, and they are particularly relevant in connection with boundary methods such as are treated in TSR #1938.

The examples given include diffraction in an unbounded medium, and static and dynamic problems in elasticity.


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# VARIATIONAL PRINCIPLES FOR PROBLEMS WITH LINEAR CONSTRAINTS. PRESCRIBED JUMPS AND CONTINUATION TYPE RESTRICTIONS

Ismael Herrera*

## 1. Introduction.

A general theory of problems subjected to linear restrictions or constraints, recently developed by the author [Herrera, 1977a, b, c; Herrera, 1978a, b; Herrera and Sabina, 1978], is presented. As general applications, a problem for which the solutions are required to satisfy prescribed jumps with respect to a given arbitrary smoothness criterion, and other ones whose solutions are subjected to restrictions of continuation type (i.e., the solutions must be such that can be continued smoothly into solutions of given equations in a neighboring region) are formulated abstractly. General variational principles for such problems are obtained. In addition, it is shown that the set of functions that can be continued in the manner explained above constitutes, generally, a completely regular subspace in the sense here defined.

These results have a bearing on boundary methods that are being developed at present [Heise, 1978; Sabina, Herrera and England, 1978; Sanchez-Sesma and Rosenblueth, 1978; Kupradze et al., 1976] for treating numerically partial differential equations associated with many problems of Science and Engineering.

Variational principles for problems formulated in discontinuous fields and with prescribed jump conditions are useful in numerical applications, and a review of such work was presented by Nemat-Nasser [1972a, b]. This kind of principles have been derived, up to now, in an 'ad hoc' manner, for each particular application. Here, general formulas are derived which can be applied whenever the smoothness criterion possesses certain properties which characterize what is here called a completely regular smoothness condition. It is shown that many of the smoothness criteria occurring in mathematical physics are completely regular. The examples given include a two-phase system, in which part of the region considered is occupied by an inviscid fluid and the rest by an elastic solid. This situation occurs, for example, when carrying out soil-structure interaction analyses of a filled dam with its foundation.

Problems subjected to constraints of the continuation type considered here, are receiving much attention in connection with the development of "boundary element methods" [Cruse and Rizzo, 1975; Brebbia, 1978]. These are numerical schemes which can be applied when a general solution is known in part of the region on which the problem is formulated and permit reducing the area covered by the numerical nets, and sometimes the dimensionality of the problem. Generally, when applying those methods auxiliary boundaries are introduced on which the sought solutions are required to be such that can be continued smoothly into neighboring regions, as solutions of the partial differential equations considered. In diffraction problems, for example, the region in which the general analytical solution is known may be a half-space [Sabina, Herrera and England, 1978]. When part of the region is treated numerically, it has interest to match this part with the rest efficiently and this can be done using variational principles [see for example, Mei and Chen, 1976]. Variational principles of this type when applied to diffraction problems, which are formulated in an unbounded region, have the interesting property that the corresponding functionals involve a bounded region only [Mei and Chen, 1976].

As mentioned previously, it is here shown that constraints of continuation type are, in general, completely regular in the sense here defined. When a constraint has this property a stronger form of the general variational principles

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developed in this paper holds. Completely regular constraints can be characterized by the fact that an antisymmetric bilinear functional vanishes in a denumerable set of functions, here called 'connectivity basis'. An additional advantage of completely regular constraints is that a general method for constructing connectivity bases which are independent of the regions considered has just been developed [Herrera and Sabina, 1978].

Most frequently, boundary methods have been formulated by means of integral equations based on Maxwell Betti's formula [Brebbia, 1978; Cruse and Rizzo, 1975; Cruse, 1974; Rizzo, 1967]; however, there are alternatives. In very general terms, one can say that the general solution which is used for the formulation of boundary methods, can be given as a family of solutions which may depend on a continuous parameter or alternatively, on a discrete parameter. In the first case the family is non-denumerable and it is usually prescribed by means of singular solutions or Green's functions, and the sought solution is constructed by means of integral representations. This type of representation is not limited to the standard Maxwell Betti's formula and Heise [1978] has discussed some of the alternatives which he calls singularity method and attributes to Rieder [1962, 1969]. Usually the boundary of the region represents the domain of definition of the given function in the integral equation as well as the sought solution, but other approaches have been considered by Kupradze [1965; Kupradze et al., 1976] who developed an integral equation for which the solution is defined on the boundary, while the data are prescribed on a curve which encloses the boundary, and by Oliveira [1968], who developed a method that in certain respects is a counterpart of Kupradze's. *A similar, but simpler procedure* has been recently applied by Sanchez-Sesma and Rosenblueth [1978].

When a denumerable family of solutions is used one is lead to series representations, *or more generally*, as Millar [1973] has pointed out, to a sequence



of least-squares approximations. The series expansion method has been used extensively in acoustics and electromagnetic field computations [Bates, 1975]. Recently, Mei and Chen [1976] have treated a problem in the theory of linearized free-surface flows, that can be recognized as an application of the series expansion method. In seismology a diffraction problem has just been solved [Sabina, Herrera and England, 1978] using a sequence of least-squares approximations in terms of a denumerable basic set of solutions.

Although the alternatives have numerical advantages (for example, they lead to non-singular equations), and have shown to be effective in many cases, there are many questions that are not well understood. Oliveira [1968], for example, presented a theorem stating the kind of conditions that boundary values must fulfill in order for an integral representation, of the kind he considered, to be valid; these were very severe conditions that many problems of technical interest do not fulfill. However, Oliveira [1968] himself showed that such problems can be solved successfully, anyway. A survey of problems associated with the use of series expansions *in acoustics and electromagnetic field computations* has been presented by Bates [1975]; many difficulties in this field are related with what is known as "Rayleigh hypothesis". However, Millar [1973], suggests that Rayleigh hypothesis can be avoided altogether if a different point of view is adopted.

In general, there are two theoretical questions which aquire great practical importance in specific applications; conditions under which a basic set of functions is complete and conditions which assure the convergence of the approximating procedure.

The fact that constraints of analytic continuation type are completely regular, which as previously mentioned is proved here, seems to be a promising tool to discuss these matters [Herrera, 1978b]. Indeed, completely regular

constraints can be characterized by connectivity bases, here defined, and a general method for constructing such bases has just been developed [Herrera and Sabina, 1978]. Furthermore, a procedure has been suggested for relating the notion of connectivity basis with that of Hilbert space basis [Herrera, 1978b]. When this is possible, a connectivity basis becomes a Hilbert space basis and the completeness of the basic set of functions is established. Once this has been shown it is a relatively simple matter to give criteria for choosing the coefficients of the linear combinations, which assure uniform convergence in the exterior domain, in a manner similar to what can be done in electromagnetic field problems [Millar, 1973]. The latter is related to a procedure presented by Kantorovich and Krylov [1964, pp. 44-68], in connection with two-dimensional problems for Laplace equation.

This paper is mainly concerned with the development of variational principles, although some of the basic notions and results on completely regular constraints are included; this was required in order to develop the subject matter more systematically. Some of the implications that the theory has on the foundations of boundary methods have been advanced [Herrera and Sabina, 1978; Herrera, 1978b], but a more complete discussion is being prepared.

As in previous work by the author [Herrera, 1974; Herrera and Bielak, 1976; Herrera and Sewell, 1978], functional valued operators are used systematically, because they have demonstrated to be quite suitable for the formulation of variational principles. Some of the results of the theory reported here, suggest that such operators are also valuable in the discussion of more general questions related to differential and integral equations; indeed, functional valued operators supply a very flexible language which permits treating general problems with simplicity, clarity and rigor. In this respect, the author hopes that this paper will estimulate more extensive use of Functional Analysis to



treat questions relevant in specific applications, because it shows that notions of a relatively elementary nature, and therefore within the grasp of a larger audience, can be used to achieve those desired features.

In Section 2, functional valued operators and the general problem with linear constraints, are introduced; regular and completely regular constraints are also defined there. Finally, the notion of connectivity basis is given.

In Section 3, canonical decompositions of a linear space $D$, are defined and it is shown how they are associated to problems with linear restrictions.

In Section 4, the notion of decomposition of an antisymmetric operator $A$, is introduced. A one to one correspondence between operators that decompose $A$ and canonical decompositions of $D$ is established.

Many boundary value problems can be cast within the frame-work of the abstract problem with linear constraints here considered. However, the main application given is in the discussion of what is called the problem of connecting and a related problem with linear restrictions of continuation type. The problem of connecting is an abstract version of a problem posed on a region such as $R \cup E$ in Figure 1, where $R$ and $E$ are neighboring sub-regions, subjected to a prescribed smoothness criterion across the common boundary. In applications, such problem corresponds to a problem formulated in discontinuous fields and with prescribed jump conditions.

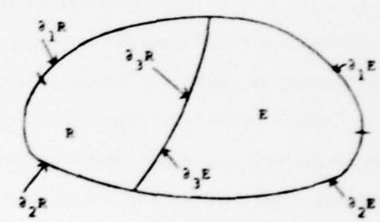


Figure 1

In Section 5, the problem of connecting is introduced and it is shown that the existence of solution for this problem grants that the set of functions that can be extended smoothly across the common boundary into solutions of the homogeneous equations on E, constitutes a linear subspace which is completely regular for the equations on R. This result is relevant for boundary methods, because the set of functions that can be so extended is characterized by the general solutions mentioned previously, which are assumed to be known beforehand in order to apply those methods.

In Section 6, general variational principles for problems with linear restrictions are formulated. Two types of results are given; one which is shown to be useful for the formulation of variational principles for problems with prescribed jump conditions and other one, which is useful in problems subjected to restrictions of analytic continuation type.

Finally, in Section 7, applications are made to potential theory and reduced wave equation, heat and wave equations. Applications to Elasticity are explained for static, periodic and dynamical problems. Also, an application to a two-phase problem is considered, in which region R (Figure 1) is occupied by an inviscid liquid, as when a dam is filled, while in E there is an elastic solid. An application to the linearized theory of free surface flows has been given previously [Herrera, 1977a].

Some of the theorems take as an assumption, the existence of solution of the abstract problems considered. In specific applications this hypothesis requires taking the linear space on which the operators are defined, so as to satisfy it. There are tratises available which discuss thoroughly questions of existence of solutions for partial differential equations [Lions and Magenes, 1968; see also, Babuska and Aziz, 1972] and therefore, we have preferred not to



discuss such matters in this paper.

The terminology of the theory has been revised; the problem with linear restrictions had been called in previous papers, problem of diffraction. Regular and completely regular subspaces, were called before, connectivity and complete connectivity conditions, respectively. It was felt that these changes were necessary because the former terminology had been suggested by specific applications, and apparently, was misleading at the more general level that the theory has achieved.

## 2. Problems with linear restrictions.

In what follows $F$ will be the field of real, or alternatively, of complex numbers. Let $D$ be a linear space and $D^*$ its algebraic dual; i.e. $D^*$ is the set of linear functionals defined on $D$. With the usual algebraic structure, $D^*$ is itself a linear space. In this paper attention will be restricted to operators $P:D \to D^*$ which are linear. The value $P(u) \in D^*$ of $P$ at $u \in D$, is a linear functional. Write $(P(u),v) \in F$ for the value of the functional $P(u) \in D^*$ at $v \in D$. When $P$ is linear, it is customary to drop the parenthesis in $P(u)$, and in this case the operator $P:D \to D^*$ is uniquely determined by the bilinear functional $(Pu,v)$. In this case, the adjoint operator $P^*:D \to D^*$ always exists and it is defined by means of the transposed bilinear functional $(Pv,u)$. Attention will be restricted to linear operators $P:D \to D^*$.

There are many problems that can be cast in the following framework.

<u>Definition 2.1.</u> <u>Consider</u> $P:D \to D^*$ <u>and a subspace</u> $I \subset D$. <u>Given</u> $U \in D$ <u>and</u> $V \in D$, <u>an element</u> $u \in D$ <u>is said to be a solution of the problem with linear restrictions or constraints, when</u>

$$Pu = PU \quad \text{and} \quad u - V \in I . \tag{2.1}$$

As an example, consider the operator $P: D \to D^*$ defined by

$$(Pu,v) = \int_R v \nabla^2 u \, d\underset{\sim}{x} \tag{2.2}$$

where region $R$ is illustrated in Figure 2. There are many ways in which $D$ can be taken, because it is only required to be a linear space without any further structure. For definiteness, one may think of $D$ as being the Sobolev space $H^s(R)$; $s \geq 2$ [Babuska and Aziz, 1972]. Define the linear subspace $I \subset D$ by

$$I = \{u \in D \mid u = 0, \text{ on } \partial R\}. \tag{2.3}$$

Then, problem (2.1) is Poisson's equation

$$\nabla^2 u = \nabla^2 U = f_R ; \quad \text{on } R \tag{2.4a}$$

subjected to boundary conditions of Dirichlet type

$$u = V = f_{\partial R} ; \quad \text{on } \partial R . \tag{2.4b}$$



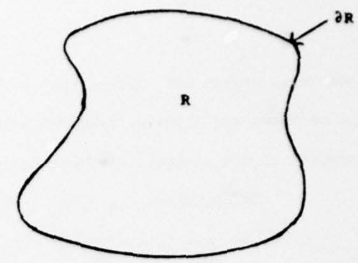


Figure 2

In the formulation given here, the functions $f_R$ and $f_{\partial R}$ may be defined, in corresponding domains by means of equations (2.4), when $U \in D$ and $V \in D$, are given. The usual practice, however, is to give functions $f_R$ and $f_{\partial R}$ on $R$ and $\partial R$, respectively, as data of the problem. When this is the case $U \in D$ and $V \in D$ can be thought of as particular solutions of (2.4a) and (2.4b), respectively. However, in most cases, the theory can be applied without actually constructing each $U$ and $V$, because all that is required is to define the linear functional $f = PU$ occurring in Equation (2.1), and this can be done without constructing $U$. Indeed, in the above example, Equation (2.2) yields

$$\langle f,v \rangle = \langle PU,v \rangle = \int_R v f_R \, d\underset{\sim}{x}$$

which only requires $f_R$ to be given. However, carrying out the general development taking as data of the problems functions such as $U \in D$ and $V \in D$, gives notational advantages and simplifies the discussions, as will become apparent in what follows.

As mentioned previously, given $P: D \to D^*$, its adjoint $P^*:D \to D^*$ always exists and it is possible to define $A:D \to D^*$ by

$$A = P - P^* . \tag{2.5}$$

The null subspace $N_A$ of $A$ will be denoted by

$$N = \{u \in D \mid Au = 0\}. \tag{2.6}$$

Definition 2.2. A subspace $I \subset D$ is said to be regular for $P$, when

a). $I \subset D$ is a commutative subspace of $P$; i.e.

$$(Au,v) = 0 \quad \forall u,v \in I \tag{2.7}$$

b). $$N \subset I \tag{2.8}$$

Regular subspaces frequently have the following additional property

c). For every $u \in D$, one has

$$(Au,v) = 0 \quad \forall v \in I \Rightarrow u \in I . \tag{2.9}$$

A regular subspace possessing property c) will be said to be completely regular for $P$.

To illustrate this notion, it can be seen that in the previous example $A : D \to D^*$ is given by

$$(Au,v) = \int_{\partial R} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n}\right) d\underline{x} \tag{2.10a}$$

and

$$N = \{u \in D \mid u = \partial u/\partial n = 0 ; \text{ on } \partial R\} . \tag{2.10b}$$

Therefore, $I \subset D$ as defined by equation (2.3) is a regular subspace for $P$; even more, it is completely regular.

Subspaces that are completely regular for $P$, can be characterized in a simple manner.

Lemma 2.1. Let $I \subset D$ be a linear subspace. Then $I$ is completely regular for the operator $P$, if and only if, for every $u \in D$ one has

$$(Au,v) = 0 \ \forall v \in I \Leftrightarrow u \in I \tag{2.11}$$

Proof. Observe that condition (2.11) is the conjuction of properties a) and c). Thus, it is enough to prove that when $I \subset D$ satisfies (2.11), $N \subset I$. This is immediate, because any $u \in N$ satisfies the premise in (2.11).

With every linear operator $P : D \to D^*$, it is possible to associate a subspace $I_p$ that is regular for $P$. It is defined by

$$I_p = N + N_p \tag{2.12}$$

where $N_p$ is the null subspace of $P$. The corresponding result is given next.



Lemma 2.2. The linear space $I_p$ defined by equation (2.12) is a regular subspace for $P$.

Proof. Condition (2.8) is clearly satisfied by $I_p$. In order to show that (2.7) is also satisfied, given any $u \in I_p$ and $v \in I_p$, write $u = u_p + u_N$ and $v = v_p + v_N$, where $u_p, v_p \in N_p$ while $u_N, v_N \in N$. Then

$$(Au,v) = (Au_p,v_p) = (Pu_p,v_p) - (Pv_p,u_p) = 0 . \tag{2.13}$$

In view of the fact that $N$ is a linear subspace of $D$, it is possible to consider the quotient spaces $\mathcal{D} = D/N$, $\mathcal{I} = I/N$ and $\mathcal{I}_p = I_p/N$. The elements of these spaces are cosets. The space $\mathcal{D}$ will be referred to as the reduced space; in applications to boundary value problems the elements of $\mathcal{D}$ are characterized by boundary values of the functions of the corresponding cosets.

For the operator $P : D \to D^*$ given by (2.2), $N$ is given by (2.10b) and therefore, each coset of $\mathcal{D} = D/N$ is characterized by a pair of functions $(u, \partial u/\partial n)$ defined on $\partial R$.

Definition 2.3. The problem with linear restrictions (2.1), is said to satisfy

a). Existence, when there is at least one solution for every $U \in D$ and $V \in D$;

b). Uniqueness, when $U = 0$ and $V = 0 \Rightarrow u = 0$;

c). Almost uniqueness, when

$$U = 0 \ \& \ V = 0 \Rightarrow u \in N .$$

By a reduced solution or boundary solution, it is meant an element $u \in \mathcal{D} = D/N$ such that $u - U^c \in \mathcal{I}_p$ while $u - V^c \in \mathcal{I}$ where $U^c, V^c \in \mathcal{D}$ stand for the cosets associated with $U$ and $V$, respectively.

In applications to boundary problems almost uniqueness corresponds to uniqueness of suitable boundary values. For example, when $N$ is given by (2.10b), the boundary values $u$ and $\partial u/\partial n$ are unique if almost uniqueness is satisfied.

The case when $V = 0$ in problem (2.1), will be called the basic problem. The properties given in Definition 2.3 depend on corresponding properties of the basic problem, only.

Lemma 2.3. The problem with linear restrictions (2.1) satisfies existence, uniqueness or almost uniqueness, respectively, if and only if, the basic problem enjoys corresponding properties.

Proof. The proof follows from the fact that if $w \in D$ is defined by $w = u - V$, with $V \in D$ fixed, then

$$Pu = PU \text{ and } u - V \in I \Leftrightarrow Pw = P(U - V) \text{ and } w \in I . \qquad (2.14)$$

In many applications it is possible and useful to replace the linear subspace $I$, occurring in the premise of (2.9), by one of its proper subsets.

Definition 2.4. Assume $I \subset D$ is a completely regular subspace for $P : D \to D^*$. Let $B \subset I$ be such that for any finite subset $\{w_1, \dots, w_n\} \subset B$, the functionals $\{Aw_1, Aw_2, \dots, Aw_n\}$ are linearly independent. If, for every $u \in D$

$$\langle Au, w \rangle = 0 \;\; \forall w \in B \Rightarrow u \in I \qquad (2.15)$$

then, $B$ is said to be a connectivity basis for $I$.

It is easily seen that a regular subspace is completely regular, if and only if, it possesses a connectivity basis. A connectivity basis may be denumerable or non-denumerable. A procedure to construct connectivity bases, applicable to many problems has been given by Herrera and Sabina [1978].

There is a very straight-forward result that will be used when formulating variational principles in Section 6. Let $S : D \to D^*$ be symmetric and $f \in D^*$; then,

$$Pu = f \Leftrightarrow \Omega'(u) = 0 \qquad (2.16)$$

where

$$\Omega(u) = \frac{1}{2} \langle Su, u \rangle - \langle f, u \rangle . \qquad (2.17)$$



Here the derivative $\Omega'$ of $\Omega : D \to F$ is taken in the sense of additive Gateaux variation [Nashed, 1971], which is probably the weakest definition of derivative. Relation (2.16) was given in [Herrera, 1974], has been used in previous work [Herrera and Bielak, 1976; Herrera and Sewell, 1978] and a related result has been given by Oden and Reddy [1976]; it follows from the fact that when $S$ is symmetric

$$\Omega'(u) = Su - f . \qquad (2.18)$$

## 3. On the occurrence of canonical decompositions.

In this section it will be seen that there is frequently associated a pair of completely regular subspaces with the problem with linear restrictions (2.1).

Definition 3.1. Let $I_1 \subset D$ and $I_2 \subset D$ be two completely regular subspaces for $P$. Then the pair $(I_1, I_2)$ is said to constitute a canonical decomposition of $D$, with respect to $P$, when

$$I_1 + I_2 = D \text{ and } I_1 \cap I_2 = N \quad . \tag{3.1}$$

Clearly, a pair $(I_1, I_2)$ of completely regular subspaces for $P$, is a canonical decomposition of $D$, if and only if, every $u \in D$ can be written as

$$u = u_1 + u_2 \quad ; \quad u_1 \in I_1,\ u_2 \in I_2 \tag{3.2}$$

and this representation is almost unique in the sense that $u_1 - u_1' \in N$ and $u_2 - u_2' \in N$ whenever $u_1', u_2'$ is any other pair satisfying (3.2).

Going back to the example considered in Section 2, a canonical decomposition $(I_1, I_2)$ of $D$, can be constructed by taking $I_1$ as the subspace given by equation (2.3) and

$$I_2 = \{u \in D \mid \partial u/\partial n = 0, \text{ on } \partial R\} \quad . \tag{3.3}$$

The interest of canonical decompositions springs from the fact that given a subspace $I \subset D$, which is regular for $P$, under very general assumptions, the pair $(I, I_p)$ constitutes a canonical decomposition of $D$. The following discussion will be oriented to prove this fact.

Lemma 3.1. Let $I \subset D$ be a regular subspace for $P$. Assume the basic problem satisfies existence. Then, for every $u \in I_p$, we have

$$(Au, w) = 0 \ \forall w \in I \Rightarrow u \in N \quad . \tag{3.4}$$

Proof. In view of (2.12), every $u \in I_p$, can be written as $u = u_p + u_N$, with $u_p \in N_p$ and $u_N \in N$. Given any $W \in D$, take $w \in I$ such that $Pw = PW$; this is possible because the basic problem satisfies existence. Then

$$(Au, W) = (Au_p, W) = -(PW, u_p)$$

$$= -(Pw, u_p) = (Au_p, w) = (Au, w) \quad . \tag{3.5}$$

Therefore,

$$(Au, w) = 0 \ \forall w \in I \Rightarrow (Au, W) = 0 \ \forall W \in D$$

$$\Rightarrow u \in N \quad . \tag{3.6}$$

Corollary 3.1. If $I \subset D$ is regular for $P$ and the basic problem satisfies existence, then

$$I \cap I_p = N \tag{3.7}$$

and the solution of the problem with linear constraints is almost unique.

Proof. $I$ and $I_p$ are regular for $P$, so that $N \subset I \cap I_p$ by the Definition 3.1 of regular subspace. Conversely, $N \supset I \cap I_p$, because the hypotheses of Lemma 3.1 are satisfied whenever $u \in I \cap I_p$. The second part of this corollary follows from the first part.

The dual of Lemma 3.1, which is obtained by interchanging the roles of $I$ and $I_p$, is also true.

Lemma 3.2. Let $I \subset D$ be a regular subspace for $P$. Assume the basic problem satisfies existence. Then, for every $u \in I$, we have

$$(Au, v) = 0 \ \forall v \in I_p \Rightarrow u \in N \quad . \tag{3.8}$$

Proof. The proof is similar to that of Lemma 3.1, but use has to be made of Lemma 2.3.

Theorem 3.1. Let $I \subset D$ be a regular subspace for $P$. If the problem with linear restrictions satisfies existence, then the pair $(I, I_p)$ constitutes a canonical decomposition of $D$. In particular $I \subset D$ and $I_p \subset D$ are completely

regular subspaces for $P$.

Proof. Assume $U \in D$, is such that

$$(AU, v) = 0 \quad \forall v \in I \ . \tag{3.9}$$

Define $w = U - u$, where $u \in I$ is such that $Pu = PU$. Therefore, $w \in I_P$ and simultaneously

$$(Aw, v) = (AU, v) - (Au, v) = 0 \quad \forall v \in I \ . \tag{3.10}$$

This shows by Lemma 3.1, that $w \in N \subset I$. Hence $U = u + w \in I$ and $I$ is completely regular. Making use of Lemma 3.2, dual of Lemma 3.1, it is possible to prove in a similar fashion, that $I_P$ is also completely regular. Corollary 3.1, shows that $I \cap I_P = N$, thus, by Definition 3.1, it remains only to prove $I + I_P = D$. This is immediate, because given $U \in D$, choose $U_1 \in I$ such that $PU_1 = PU$, which is possible because existence for the problem with linear constraints is assumed. Define $U_2 = U - U_1$, then $U = U_1 + U_2$ and $U_1 \in I$ while $U_2 \in N_P \subset I_P$.



## 4. Decompositions of A and canonical decompositions.

There is a close connection between canonical decompositions and certain classes of decompositions of the operator $A$. This section is devoted to establish such relations.

Definition 4.1. An operator $B : D \to D^*$ is said to be determined by $A$, when

$$N_B \supset N \ . \tag{4.1}$$

Here $N_B$ is the null subspace of $B$.

As an example, $B: D \to D^*$ given by

$$(Bu, v) = \int_{\partial R} v \frac{\partial u}{\partial n} d\underset{\sim}{x} \tag{4.2}$$

is determined by $A$, as given by (2.10a).

Definition 4.2. Given operators $P: D \to D^*$ and $Q: D \to D^*$, one says that $P$ and $Q$ can be varied independently when for every $U \in D$ and $V \in D$, there exists $u \in D$ such that

$$Pu = PV \quad \text{and} \quad Qu = QV \ . \tag{4.3}$$

The proof of the following lemma is straight-forward.

Lemma 4.1. Let $P : D \to D^*$ and $Q : D \to D^*$ be linear operators. Then the following assertions are equivalent

a). $P$ and $Q$ can be varied independently.

b). For every $U \in D$, $\exists u \in D \ni Pu = PU; \ Qu = 0$ . (4.4)

c). For every $V \in D$, $\exists u \in D \ni Pu = 0 \ ; \ Qu = QV$. (4.5)

As an example, the operator $B: D \to D^*$ as given by (4.2) and $B^*: D \to D^*$

$$(B^*u, v) = \int_{\partial R} u \frac{\partial v}{\partial n} d\underset{\sim}{x} \tag{4.6}$$

can be varied independently.

Definition 4.3. An operator $B: D \to D^*$ is said to decompose $A$, when $B$ and $B^*$ can be varied independently and

$$A = B - B^* \ . \tag{4.7}$$

Applying Definition 4.3, we can say that the operator $B: D \to D^*$ defined by (4.2), decomposes $A$.

Lemma 4.2. Assume $B : D \to D^*$ decomposes $A$. Then $B$ and $B^*$ are determined by $A$.

Proof. In view of Definition 4.1, it is necessary to prove, that when $B$ decomposes $A$, one has

$$Au = 0 \Rightarrow Bu = 0 . \tag{4.8}$$

If $B$ decomposes $A$,

$$Au = 0 \Rightarrow Bu = B^*u . \tag{4.9}$$

Given any $V \in D$, choose $v \in D$ such that $Bv = 0$ and $B^*v = B^*V$. Then if $Au = 0$,

$$\langle Bu,V \rangle = \langle Bu,v \rangle = \langle B^*u,v \rangle = \langle Bv,u \rangle = 0 . \tag{4.10}$$

This shows that $Bu = 0$ because $V \in D$ is arbitrary. Hence, $B$ is determined by $A$. The fact that $B^*$ is also determined by $A$ follows from the above result when it is observed that $-B^*$ decomposes $A$ whenever $B$ does.

It is possible to establish a one to one correspondence between operators $B : D \to D^*$ that decompose $A$ and canonical decompositions of $D$.

Theorem 4.1. Assume $B : D \to D^*$ decomposes $A$, then the pair of linear subspaces $(I_1,I_2)$ given by

$$I_1 = \{u \in D | Bu = 0\} = N_B \tag{4.11a}$$

and

$$I_2 = \{u \in D | B^*u = 0\} = N_{B^*} \tag{4.11b}$$

constitutes a canonical decomposition of $D$ with respect to $P$.

Conversely, given any canonical decomposition $(I_1,I_2)$, define $B: D \to D^*$ by

$$\langle Bu,v \rangle = \langle Au_2,v_1 \rangle \tag{4.12}$$

where $u = u_1 + u_2$, $u_1 \in I_1$, $u_2 \in I_2$, and similarly for $v$. Then $B$ decomposes $A$ and satisfies (4.11). Even more, this is the only operator with these properties.



Proof. Observe that canonical decompositions must be understood as ordered pairs $(I_1,I_2)$; thus, in general $(I_1,I_2)$ is different from $(I_2,I_1)$.

To prove this Theorem, it will be first shown that when $B$ decomposes $A$, $I_1$ and $I_2$ as given by (4.11), are completely regular. This can be seen by showing that condition (2.11) of Lemma 2.1 is satisfied by $I_1$ and $I_2$. Now

$$\langle Au,v \rangle = \langle Bu,v \rangle - \langle Bv,u \rangle = 0 , \ \forall u,v \in I_1 . \tag{4.13}$$

To prove the converse implication in (2.11), observe that given any $V \in D$, it is possible to choose $v \in D$ such that $Bv = 0$ (i.e. $v \in I_1$) and simultaneously $B^*v = B^*V$, because $B$ and $B^*$ can be varied independently. With this choice of $v \in I_1$

$$\langle Bu,V \rangle = \langle B^*v,u \rangle = - \langle Au,v \rangle. \tag{4.14}$$

This shows that $\langle Au,v \rangle = 0 \ \forall v \in I_1$ implies $u \in I_1$ because $V \in D$ is arbitrary in (4.14). Hence, $I_1$ is completely regular. A similar argument proves the corresponding result for $I_2$.

In order to show that $(I_1,I_2)$ is a canonical decomposition of $D$, it remains to prove that $I_1 \cap I_2 = N$ and $I_1 + I_2 = D$. Clearly, $I_1 \cap I_2 \supset N$ in view of Lemma 4.2. Conversely, $N \supset I_1 \cap I_2 = N_B \cap N_{B^*}$, because $A = B-B^*$. Given $u \in D$ choose $u_1 \in D$ so that $Bu_1 = 0$ while $B^*u_1 = B^*u$, which is possible because $B$ and $B^*$ can be varied independently. Define $u_2 = u - u_1$, then $B^*u_2 = 0$ and $u = u_1 + u_2$; this shows that $D = I_1 + I_2$ because $u_1 \in I_1$ while $u_2 \in I_2$. The proof of the first part of Theorem 4.1 is now complete.

To prove the second part, let $(I_1,I_2)$ be an arbitrary canonical decomposition of $D$. Given any $u,v \in D$, take $u_1,v_1 \in I_1$ and $u_2,v_2 \in I_2$ as the components of the almost unique representations of $u$ and $v$, corresponding to the canonical decomposition $(I_1,I_2)$ of $D$. Then, the operator $B : D \to D^*$ given by (4.12) is unambiguously defined. The commutative property (2.7) of regular subspaces imply that

$$(Au,v) = (Au_2,v_1) - (Av_2,u_1) . \tag{4.15}$$

This shows $A = B - B^*$. To prove that $B$ and $B^*$ can be varied independently, let $U \in D$ and $V \in D$ be given, then $u = V_1 + U_2$ satisfies $Bu = BU$ and $B^*u = B^*V$. Thus, $B$ decomposes $A$. To see that equations (4.11) are satisfied, observe that $Bu = 0$ implies

$$(Bu,v) = (Au_2,v) = 0 \ \forall v \in D . \tag{4.16}$$

Hence $u_2 \in N$ and therefore $u = u_1 + u_2 \in I_1$. Conversely, if $u \in I_1$, then $u_2 \in N$ and $Bu = 0$ by virtue of (4.12). This completes the proof of (4.11a); the proof of (4.11b) is similar.

To prove uniqueness, it will be shown that equation (4.12) is necessarily satisfied by any such $B$. Assume $B : D \to D^*$ is such that $A = B - B^*$ and it satisfies (4.11). Then $Bu_1 = 0$, $\forall u_1 \in I_1$ and $B^*u_2 = 0$, $\forall u_2 \in I_2$; therefore

$$(Au_2,v_1) = (Au_2,v) = (Bu_2,v) = (Bu,v) . \tag{4.17}$$



5. <u>The problem of connecting</u>.

There are many problems that can be formulated as problems with linear restrictions; a very general example is the problem of connecting.

Although the formulation to be presented is an abstract one, it is motivated by a specific situation. Assume there are two neighboring regions $R$ and $E$ (Figure 1) with boundaries $\partial R$ and $\partial E$, respectively. By reasons that will become apparent in some of the examples to be given, the common boundary between $R$ and $E$ will be denoted $\partial_3 R = \partial_3 E$. The general problem is to find solutions to specific partial differential equations on $R \cup E$ subjected to a given smoothness criterion across the connecting boundary $\partial_3 R = \partial_3 E$. Problems of this kind occur frequently in applications; the smoothness criterion may be in potential theory, for example, the $u$ and $\partial u/\partial n$ be continuous across $\partial_3 R$, or in Elasticity, that displacements and tractions be continuous across that part of the boundary, but more complicated criteria may be included in the theory.

Let $\hat{D}$ be a linear space and $\hat{P}:\hat{D} \to \hat{D}^*$ a functional valued operator defined on that space. Here again, $\hat{P}$ is assumed to be linear; in addition, $\hat{D} = D_R \oplus D_E$ where $D_R$ and $D_E$ are two linear spaces. Elements $\hat{u} \in \hat{D}$ will be thought as pairs $(u_R, u_E)$, where $u_R \in D_R$ and $u_E \in D_E$. The space $\hat{D}^*$ is the algebraic dual of $\hat{D}$ and the operator $\hat{P}$ is assumed to have the additive property

$$(\hat{P}\hat{u}, \hat{v}) = (\hat{P}u_R, v_R) + (\hat{P}u_E, v_E) \tag{5.1}$$

for every $\hat{u} = (u_R, u_E)$, $\hat{v} = (v_R, v_E)$. If the operators $\hat{P}_R:\hat{D} \to \hat{D}^*$ and $\hat{P}_E:\hat{D} \to \hat{D}^*$ are defined by

$$(\hat{P}_R\hat{u}, \hat{v}) = (\hat{P}u_R, v_R) \ ; \ (\hat{P}_E\hat{u}, \hat{v}) = (\hat{P}u_E, v_E) \tag{5.2}$$

then

$$\hat{P} = \hat{P}_R + \hat{P}_E \quad . \tag{5.3}$$

Operators $P_R : D_R \to D_R^*$ and $P_E : D_E \to D_E^*$ can also be defined; they are given by

$$(P_R u_R, v_R) = (\hat{P}u_R, v_R) \; ; \; (P_E u_E, v_E) = (\hat{P}u_E, v_E) \quad . \tag{5.4}$$

Then

$$(\hat{P}\hat{u}, \hat{v}) = (P_R u_R, v_R) + (P_E u_E, v_E) \quad . \tag{5.5}$$

Using these operators, the following can be defined

$$\hat{A} = \hat{P} - \hat{P}^*, \; \hat{A}_R = \hat{P}_R - \hat{P}_R^* \; ; \; A_R = P_R - P_R^* \; ; \; \hat{A}_E = \hat{P}_E - \hat{P}_E^* \; ; \; A_E = P_E - P_E^* \; . \tag{5.6}$$

They satisfy

$$\hat{A} = \hat{A}_R + \hat{A}_E \tag{5.7}$$

and

$$(\hat{A}\hat{u}, \hat{v}) = (A_R u_R, v_R) + (A_E u_E, v_E) \quad . \tag{5.8}$$

The null subspaces of $\hat{A}$, $\hat{A}_R$, $\hat{A}_E$, $A_R$ and $A_E$ will be denoted by $\hat{N}$, $\hat{N}_R$, $\hat{A}_E$, $N_R$ and $N_E$, respectively. The relation

$$\hat{N} = N_R \oplus N_E \tag{5.9}$$

will be used later; it is equivalent to

$$\hat{u} = (u_R, u_E) \in \hat{N} \Leftrightarrow u_R \in N_R \text{ and } u_E \in N_E \quad . \tag{5.10}$$

This latter relation follows from (5.8).

The general problem to be considered will be one with linear restrictions, where the linear subspace $\hat{S} \subset \hat{D}$ specifying the linear restriction will be assumed to satisfy special conditions. Elements $\hat{u} = (u_E, u_R) \in \hat{S}$ will be called smooth; when $\hat{u} = (u_E, u_R)$ is smooth, $u_E \in D_E$ and $u_R \in D_R$ will be said to be smooth extensions of each other.



Definition 5.1. Let $\hat{S} \subset \hat{D} = D_R \oplus D_E$ be a linear subspace. Then $\hat{S}$ will be said to be a smoothness condition or relation if every $u_R \in D_R$ possesses at least one smooth extension $u_E \in D_E$ and conversely.

Definition 5.2. Given a smoothness relation $\hat{S} \subset \hat{D}$ and elements $\hat{U} \in \hat{D}$, $\hat{V} \in \hat{D}$, the problem of connecting consists in finding an element $\hat{u} \in \hat{D}$ such that

$$\hat{P}\hat{u} = \hat{P}\hat{U} \text{ and } \hat{u} - \hat{V} \in \hat{S} \quad . \tag{5.11}$$

Clearly, the problem of connecting is a problem with linear restrictions in the sense of Definition 2.1 and the results of previous sections are applicable. The smoothness relation $\hat{S}$ will be said to be regular and completely regular for $\hat{P}$, when as a subspace, it is regular and completely regular for $\hat{P}$, respectively.

Lemma 5.1. A smoothness condition $\hat{S} \subset \hat{D}$ is completely regular for $\hat{P}$, if and only if

$$(\hat{A}\hat{u}, \hat{v}) = (A_R u_R, v_R) + (A_E u_E, v_E) = 0 \;\; \forall \hat{v} \in \hat{S} \Leftrightarrow \hat{u} \in \hat{S} \quad . \tag{5.12}$$

Proof. This lemma follows from (2.11) and (5.8).

As an example, take $D_R = H^s(R)$ and $D_E = H^s(E)$, with $s \geq 2$. Assume each of the boundaries $\partial R$ and $\partial E$ of regions $R$ and $E$ (Figure 1) is divided into three parts $\partial_i R$ and $\partial_i E$ $(i = 1,2,3)$, where $\partial_3 R = \partial_3 E$ is the common boundary between $R$ and $E$. Let $\underline{n}$ be the unit normal vector on these boundaries, which will be taken pointing outwards from $R$ and from $E$. On the common boundary $\partial_3 R = \partial_3 E$, there are defined two unit normal vectors which have opposite senses, one associated with $R$ and the other one with $E$. Some times they will be represented by $\underline{n}_R$ and $\underline{n}_E$; more often, however, the ambiguity will be resolved by the suffix used under the integral sign.

Define $P_R : D_R \to D_R^*$ by

$$(P_R u_R, v_R) = \int_R v_R \nabla^2 u_R \, d\underline{x} + \int_{\partial_1 R} u_R \frac{\partial v_R}{\partial n} \, d\underline{x} - \int_{\partial_2 R} v_R \frac{\partial u_R}{\partial n} \, d\underline{x} \tag{5.13}$$

and let $P_E : D_E \to D_E^*$ satisfy the equation that is obtained when $R$ is replaced by $E$ in (5.13). Then

$$(\hat{A}\hat{u}, \hat{v}) = \int_{\partial_3 E} \{v_E \frac{\partial u_E}{\partial n} - u_E \frac{\partial v_E}{\partial n}\} d\underline{x} + \int_{\partial_3 R} \{v_R \frac{\partial u_R}{\partial n} - u_R \frac{\partial v_R}{\partial n}\} d\underline{x} \tag{5.14}$$

while

$$\hat{N} = \{\hat{u} \in \hat{D} | u_R = u_E = \partial u_R/\partial n = \partial u_E/\partial n = 0, \ \text{on } \partial_3 R\} \ . \tag{5.15}$$

Let

$$\hat{S} = \{\hat{u} \in \hat{D} | u_R = u_E \ ; \ \partial u_R/\partial n_R = \partial u_E/\partial n_R, \ \text{on } \partial_3 R\} \ . \tag{5.16}$$

Functions $u_R \in D_R = H^s(R)$ $(s \geq 2)$ are such that their boundary values $u_R$, $\partial u_R/\partial n$ belong to $H^{s-1/2}(\partial_3 R)$ and $H^{s-3/2}(\partial_3 R)$ respectively [see for example Babuska and Aziz, 1972]. A corresponding result holds for functions $u_E \in D_E = H^s(E)$. This shows that every $u_R \in D_R$ can be extended smoothly into a function $u_E \in D_E$, and conversely. Thus $\hat{S}$ is a smoothness relation.

In this case the problem of connecting is

$$\nabla^2 \hat{u} = \nabla^2 \hat{U} \quad , \ \text{on } R \cup E \tag{5.17a}$$

$$\hat{u} = \hat{U} \quad , \ \text{on } \partial_1 (R \cup E) \tag{5.17b}$$

$$\frac{\partial \hat{u}}{\partial n} = \frac{\partial \hat{U}}{\partial n} \quad , \ \text{on } \partial_2 (R \cup E) \tag{5.17c}$$

subjected to

$$u_E - u_R = V_E - V_R, \ \partial(u_E - u_R)/\partial n = \partial(V_E - V_R)/\partial n, \ \text{on } \partial_3 R \ . \tag{5.18}$$

When $\hat{v} = (v_R, v_E) \in \hat{S}$,



$$(A_R u_R, v_R) + (A_E u_E, v_E) = \int_{\partial_3 R} \{v_R \left(\frac{\partial u_R}{\partial n} - \frac{\partial u_E}{\partial n}\right) - (u_R - u_E) \frac{\partial v_R}{\partial n}\} d\underline{x} \tag{5.19}$$

for arbitrary $\hat{u} = (u_R, u_E) \in \hat{D}$. Using (5.19) it can be seen that condition (5.12) is satisfied by $\hat{S} \subset \hat{D}$; this shows that $\hat{S}$ is completely regular for $\hat{P}$.

Well known results about the existence of solution for boundary value problems of elliptic equations [Babuska and Aziz, 1972], can be used to show that the problem of connecting corresponding to equations (5.17) and (5.18), satisfies existence when $D_R = H^s(R)$, $D_E = H^s(E)$ and $s \geq 2$, when the boundaries of $R$ and $E$ satisfy suitable regularity assumptions.

When $\hat{S}$ is completely regular, it is easy to construct a completely regular subspace which together with $\hat{S}$ constitutes a canonical decomposition of $\hat{D}$, for the operator $\hat{P}$.

<u>Definition 5.2</u>. <u>An element</u> $\hat{u} = (u_R, u_E) \in \hat{D}$ <u>is said to have zero mean when</u> $(u_R, -u_E) \in \hat{S}$. <u>The collection of elements of</u> $\hat{D}$ <u>with zero mean will be denoted by</u> $\hat{M}$.

<u>Theorem 5.1</u>. <u>When the smoothness relation</u> $\hat{S}$ <u>is completely regular, the pair</u> $(\hat{S}, \hat{M})$ <u>constitutes a canonical decomposition of</u> $\hat{D}$.

<u>Proof</u>. In view of Definition 3.1, it is required to prove that $\hat{M}$ is completely regular for $P$ and that

$$\hat{S} \cap \hat{M} = \hat{N} \ ; \ \hat{S} + \hat{M} = \hat{D} \ . \tag{5.20}$$

Clearly, $\hat{M}$ is a linear subspace of $\hat{D}$. In addition, Lemma 5.1 and the fact that $\hat{S}$ is completely regular imply that (5.12) holds. In view of Definition 5.2, $\hat{S}$ can be replaced by $\hat{M}$ in (5.12) without altering its validity. This shows that $\hat{M}$ is completely regular for $\hat{P}$.

Assume $\hat{u} = (u_R, u_E) \in \hat{S} \cap \hat{M}$; i.e. $(u_R, u_E) \in \hat{S}$ and $(u_R, -u_E) \in \hat{S}$. Then $(u_R, 0) \in \hat{S}$, which implies

$$(A_R u_R, v_R) = 0 \quad \forall v_R \in D_R \tag{5.21}$$

by virtue of (5.12) and the fact that any $v_R$ has a smooth extension. Hence, $u_R \in N_R$. In a similar manner, it is seen that $u_E \in N_E$. Therefore, $\hat{u} \in N_R \oplus N_E = \hat{N}$, by (5.9), and the first equation in (5.20) is established. To show the second of those equations, given any $\hat{u} = (u_R, u_E) \in \hat{D}$, choose smooth extensions $u_R' \in D_R$ and $u_E' \in D_E$ of $u_E \in D_E$ and $u_R \in D_R$, respectively. Then

$$\hat{u} = \overline{u} - \frac{1}{2}[\hat{u}] \tag{5.22}$$

where $\overline{u} \in \hat{S}$ and $[\hat{u}] \in \hat{M}$ are

$$\overline{u} = \frac{1}{2}(u_R' + u_R, u_E' + u_E) \tag{5.23a}$$

$$[\hat{u}] = (u_R' - u_R, u_E' - u_E) \quad . \tag{5.23b}$$

Equation (5.22) shows that any $\hat{u} \in \hat{D}$ can be written

$$\hat{u} = \hat{u}_1 + \hat{u}_2 \; ; \; \hat{u}_1 \in \hat{S} \text{ and } \hat{u}_2 \in \hat{M} \tag{5.24}$$

with

$$\hat{u}_1 = \overline{u} \; ; \; \hat{u}_2 = -[\hat{u}]/2 \quad . \tag{5.25}$$

This establishes the second of equations (5.20), and the proof of Theorem 5.1 is complete.

The fact that the pair $(\hat{S}, \hat{M})$ constitutes a canonical decomposition of $\hat{D}$, implies that given any $\hat{u} \in \hat{D}$, the elements $\overline{u} \in \hat{S}$ and $[\hat{u}] \in \hat{M}$ are defined up to elements of $\hat{N}$; more precisely, that $\overline{u}$ as well as $[\hat{u}]$, define unique cosets of the space $\hat{D}/\hat{N}$. Elements $\overline{u}$ and $[\hat{u}]$ satisfying (5.23) will be called the average and the jump of $\hat{u}$, respectively.

By means of Theorem 4.1, it is possible now to define an operator $\hat{B}:\hat{D} \to \hat{D}^*$ that decomposes $\hat{A}$ and satisfies (4.11) with $I_1 = \hat{S}$ and $I_2 = \hat{M}$. Such operator will be denoted by $\hat{J}$ and satisfies



$$2(\hat{J}\hat{u}, \hat{v}) = 2(\hat{A}\hat{u}_2, \hat{v}_1) = -(\hat{A}[\hat{u}], \overline{v}) \tag{5.26}$$

by virtue of (4.12) and (5.25). The operator $\hat{J}:\hat{D} \to \hat{D}^*$ defined by (5.26) will be called jump operator. It characterizes $\hat{S}$ because $\hat{J}\hat{u} = 0 \Leftrightarrow \hat{u} \in \hat{S}$ (Equation 4.11a).

Equation (5.26) will be used extensively when formulating variational principles for problems with prescribed jumps in discontinuous fields, and it is worthwhile to ellaborate it further. Let $\hat{u} = \hat{u}_1 + \hat{u}_2$ ; $\hat{v} = \hat{v}_1 + \hat{v}_2$, where $\hat{u}_1 = (u_{1R}, u_{1E}) \in \hat{S}$, $\hat{u}_2 = (u_{2R}, u_{2E}) \in \hat{M}$ and similarly for $\hat{v}$. Then

$$\begin{aligned}(\hat{J}\hat{u}, \hat{v}) &= (\hat{A}\hat{u}_2, \hat{v}_1) = (A_R u_{2R}, v_{1R}) + (A_E u_{2E}, v_{1E}) \\ &= 2(A_R u_{2R}, v_{1R}) = 2(\hat{A}_R \hat{u}_2, \hat{v}_1) = 2(\hat{A}_E \hat{u}_2, \hat{v}_1)\end{aligned} \tag{5.27}$$

where (5.8), (5.12) and the Definition 5.2 of $\hat{M}$ have been used. Hence

$$(\hat{J}\hat{u}, \hat{v}) = -(\hat{A}_R[\hat{u}], \overline{v}) = -(\hat{A}_E[\hat{u}], \overline{v}) \tag{5.28}$$

by virtue of (5.25). In addition

$$(\hat{A}\hat{u}, \hat{v}) = (\hat{A}_R[\hat{v}], \overline{u}) - (\hat{A}_R[\hat{u}], \overline{v}) \tag{5.29}$$

because $\hat{A} = \hat{J} - \hat{J}^*$.

The use of formulas (5.28) and (5.29), will be illustrated applying them to the previous example. In view of (5.16), the smooth extensions $u_R' \in D_R$ and $u_E' \in D_E$ of $u_R$ and $u_E$, respectively, satisfy

$$u_R' = u_E \; ; \; \partial u_R'/\partial n_R = \partial u_E'/\partial n_R, \quad \text{on } \partial_3 R \quad . \tag{5.30}$$

In addition

$$(A_R u_R, v_R) = \int_{\partial_3 R} \left(v_R \frac{\partial u_R}{\partial n} - u_R \frac{\partial v_R}{\partial n}\right) dx \quad . \tag{5.31}$$

Applying (5.28)

$$(\hat{J}\hat{u}, \hat{v}) = -\int_{\partial_3 R} \{\langle\bar{v}\rangle_R \frac{\partial[\hat{u}]_R}{\partial n} - [\hat{u}]_R \frac{\partial\langle\bar{v}\rangle_R}{\partial n}\}dx \quad . \tag{5.32}$$

Equation (5.23) yields

$$[\hat{u}]_R = u_E - u_R \; ; \; \frac{\partial[\hat{u}]_R}{\partial n} = \frac{\partial u_E}{\partial n} - \frac{\partial u_R}{\partial n}; \quad \text{on } \partial_3 R \tag{5.33a}$$

$$\langle\hat{v}\rangle_R = \frac{1}{2}(v_E + v_R) \; ; \; \frac{\partial\langle\bar{v}\rangle_R}{\partial n} = \frac{1}{2}\left(\frac{\partial v_E}{\partial n} + \frac{\partial v_R}{\partial n}\right), \quad \text{on } \partial_3 R \tag{5.33b}$$

by virtue (5.16). Equation (5.32) can be simplified if the component to be used is indicated by the index under the integral sign; thus

$$(\hat{J}\hat{u}, \hat{v}) = \int_{\partial_3 R} \{[\hat{u}]\frac{\partial\bar{v}}{\partial n} - \bar{v}[\frac{\partial\hat{u}}{\partial n}]\}dx$$

$$= \int_{\partial_3 E} \{[\hat{u}]\frac{\partial\bar{v}}{\partial n} - \bar{v}[\frac{\partial\hat{u}}{\partial n}]\}dx \tag{5.34}$$

where $[\partial\hat{u}/\partial n]_R = \partial u_E/\partial n - \partial u_R/\partial n$, on $\partial_3 R$. The last equality in (5.34) follows from the second equation in (5.28), but can also be seen because there is a double change of signs on each term appearing in the integrals; one due to the change in the sense of the unit normal and the other one due to the change of sign of the jump of $\hat{u}$. Equation (5.29) yields

$$(\hat{A}\hat{u}, \hat{v}) = \int_{\partial_3 R} \{[\hat{u}]\frac{\partial\bar{v}}{\partial n} + \bar{u}[\frac{\partial\hat{v}}{\partial n}] - \bar{v}[\frac{\partial\hat{u}}{\partial n}] - [\hat{v}]\frac{\partial\bar{u}}{\partial n}\}dx \quad . \tag{5.35}$$

The following definition and results establish a relation between the problem of connecting and problems subjected to restrictions of continuation type.

Definition 5.3. Let $D$, $P:D \to D^*$ and a linear subspace $I \subset D$ be given. Then the problem with linear restrictions (2.1), will be said to be subjected to a constraint of continuation type, when for some $\hat{D} = D_R \oplus D_E$, $\hat{P}:\hat{D} \to \hat{D}^*$ and smoothness criterion $\hat{S} \subset \hat{D}$ : a) $D = D_R$, b) $P = P_R:D_R \to D_R^*$ and c)

$$I = \{u \in D | \exists \hat{u} = (u, u_E) \in \hat{S}, P_E u_E = 0\} \quad . \tag{5.36}$$



Theorem 5.2. Assume problem (2.1) is subjected to restrictions of continuation type and the associated smoothness condition $\hat{S}$ is regular. Then, if the associated problem of connecting satisfies existence, the linear subspace $I \subset D$ is completely regular for $P$.

Proof. Theorem 3.1 will be applied to show that $(I, I_p)$ is a canonical decomposition of $D$. Here, according to Equation 2.12, $I_p = N + N_p$. By Theorem 3.1, it is only necessary to prove that $I \subset D$ is a regular subspace for $P$ and that the problem with linear restrictions satisfies existence. Given any $u \in I$ and $v \in I$, take $u_E \in D_E$ satisfying the conditions of (5.36) and similarly $v_E \in D_E$. Then

$$(Au, v) = -(A_E u_E, v_E) = (P_E v_E, u_E) - (P_E u_E, v_E) = 0 \tag{5.37}$$

where use has been made of (5.12). The condition $N \subset I$ follows from the fact that $\hat{N} \subset \hat{S}$, using (5.9) or equivalently (5.10). This shows that $I \subset D$ is a regular subspace for $P$.

By virtue of Lemma 2.3, it remains to prove that the basic problem

$$Pu = PU \quad ; \quad u \in I \tag{5.38}$$

satisfies existence. To prove this, given $U \in D$, define $\hat{U} = (U, 0) \in \hat{D}$ and let $\hat{u} = (u, u_E)$ be a solution of the problem of connecting

$$\hat{P}\hat{u} = \hat{P}\hat{U} \quad ; \quad \hat{u} \in \hat{S} \tag{5.39}$$

Then, recalling definition (5.36), it is seen that $u \in D$ satisfies (5.38), and the proof of Theorem 5.2 is complete.

As an example, in Figure 1, the functions of $D = H^s(R)$, $(s \geq 2)$, that can be continued smoothly into function of $H^s(E)$ that are harmonic on $E$, vanish on $\partial_1 E$ and whose normal derivative vanishes on $\partial_2 E$, constitutes a completely regular subspace for $P:D \to D^*$, defined by

$$(Pu, v) = \int_R v\nabla^2 u dx + \int_{\partial_1 R} u\frac{\partial v}{\partial n}dx - \int_{\partial_2 R} v\frac{\partial u}{\partial n}dx \quad . \tag{5.40}$$

Here, the criterion of smoothness is that $u$ and $\partial u/\partial n$ are continuous across $\partial_3 R$. Such result can be extended to unbounded regions if suitable radiations conditions are imposed on the functions considered [Herrera and Sabina, 1978].



## 6. Variational Principles.

The theory developed in this paper will be used in this section to formulate two types of variational principles for problems with linear restrictions.

The first one applied when there is available a canonical decomposition $(I, I_c)$, one of whose elements is the linear subspace $I$ which specifies the restriction in problem (2.1). In this case, $P - B$, where $B: D \to D^*$ is the operator associated with the canonical decomposition by means of (4.12), is symmetric; by its use one obtains variational principles for which the variations need not be restricted. However, it must be observed that the mere existence of such canonical decomposition is not sufficient to permit the formulation of these variational principles; it is required, in addition, that the actual decomposition of every vector $u \in D$ in its components $u_1$ and $u_2$, can be carried out without difficulty, because this is necessary in order to construct $B$ by means of (4.12). Problems subjected to restrictions of continuation type, do not fulfill this requirement in spite of the fact that for them $(I, I_p)$, frequently constitutes a canonical decomposition; this can be seen by observing that to obtain the components $U_1, U_2$ of any $U \in D$ with respect to this canonical decomposition, it is essentially required to solve the problem with linear restrictions (2.1).

When the operator $B$ cannot be constructed the second type of variational principle can be applied. It is associated with the operator $2P - A$, which is always symmetric and can be used if variations are restricted to be in the regular subspace $I$; the results are enhanced when the subspace is completely regular, as is often the case.

Applications are made to the problem of connecting, for which the construction of $B$ (the jump operator) is possible, as shown in Section 5, and to problems with restrictions of continuation type, for which, as already

mentioned, such construction is not possible and the operator $2P - A$ has to be used.

The following lemmas lead to the desired variational principles.

Lemma 6.1. Let $I \subset D$ be a completely regular subspace for $P$, then given $U \in D$ and $V \in D$, an element $u \in D$ is solution of the problem with linear constraints (2.1), if and only if

$$Pu = PU \tag{6.1}$$

and

$$(A(u-V), v) = 0 \quad \forall v \in I \ . \tag{6.2}$$

When $I$ is regular, but not completely regular, the above assertion holds for elements $u \in V + I$.

Proof. The mere regularity of $I \subset D$, is enough to guarantee that Equation (2.1) implies (6.1) and (6.2). When, in addition $I \subset D$ is completely regular, conversely, (6.2) implies that $u - V \in I$; hence, Equation (2.1) follows from (6.1) and (6.2), in this case. The second part of the lemma is now straight-forward.

Lemma 6.2. Assume $(I, I_c)$ constitutes a canonical decomposition of $D$ with respect to $P$, and let $B:D \to D^*$ be defined by (4.12), taking $u_2$ and $v_1$ as components of vectors on $(I, I_c)$. Then $u \in D$ is a solution of the problem with linear constraints (2.1), if and only if

$$Pu = PU \ \text{and} \ Bu = BV \ . \tag{6.3}$$

Proof. By Theorem 4.1, Equation (4.11a), $u - V \in I$ if and only if $B(u-V) = 0$.

Definition 6.1. An operator $P:D \to D^*$ is said to be formally symmetric when for every $u \in D$

$$(Pu, v) = 0 \ \forall v \in N \Rightarrow Pu = 0 \ . \tag{6.4}$$



It is customary to call a differential operator $L$, formally symmetric, when

$$\int_R \{vLu - uLv\} d\underline{x} = \text{boundary terms} \ . \tag{6.5}$$

To such an operator one can associate a $P:D \to D^*$ which is formally symmetric in the sense of Definition 6.1 by means of

$$(Pu, v) = \int_R vLud\underline{x} \ . \tag{6.6}$$

As an example, the operator associated by means of (6.6) to the Laplacian, is formally symmetric in the sense of Definition 6.1. Indeed, in this case $P:D \to D^*$ is given by equation (2.2) and the null subspace (Equation 2.10b) is the set of functions which together with their normal derivatives, vanish on the boundary. Property (6.4), in this case, amounts to the so called, fundamental lemma of calculus of variations.

Lemma 6.3. Assume $P:D \to D^*$ is formally symmetric and $I \subset D$ is regular for $P$. Then

a). (6.1) and (6.2) hold simultaneously if and only if

$$((2P - A)u, v) = (2PU - AV, v) \quad \forall v \in I \tag{6.7}$$

b). Equation (6.3) holds, if and only if

$$(P - B)u = PU - BV \ . \tag{6.8}$$

Proof. Rearranging, equation (6.7) becomes

$$(2P(u-U), v) = (A(u-V), v) \quad \forall v \in I \ . \tag{6.9}$$

Clearly, (6.1) and (6.2) imply (6.9). Conversely, (6.9) implies

$$(2P(u-U), v) = 0 \quad \forall v \in N \subset I \tag{6.10}$$

which in turn implies (6.1), because $P$ is formally symmetric. Once this has been shown, (6.9) reduces to (6.2). This proves a).

Equation (6.8) can be obtained subtracting one of equations (6.3) from the other. Conversely, (6.8) implies

$$(P(u-U), v) = (B(u-V), v) = 0 \quad \forall v \in N \subset D \tag{6.11}$$

because according to Lemma 4.2, $B^*$ is determined by $A$ (i.e. $N_{B^*} \supset N$). The first of equations (6.3) follows from (6.11), because $P$ is formally symmetric. Once that equation has been proved, (6.8) reduces to the second equation in (6.3).

Theorem 6.1. Assume $P{:}D \to D^*$ is formally symmetric and $(I, I_c)$ constitutes a canonical decomposition of $D$. Then $u \in D$ is a solution of the problem with linear restrictions (2.1), if and only if

$$\Omega'(u) = 0 \tag{6.12}$$

where

$$\Omega(u) = \frac{1}{2}((P-B)u, u) - (PU - BV, u) \quad . \tag{6.13}$$

Here $B{:}D \to D^*$ is the operator associated with $(I, I_c)$ by means of (4.12).

Proof. Recall that $P-P^* = A = B-B^*$; hence, $P-B$ is symmetric. Applying (2.17) to this symmetric operator, Theorem 6.1 follows from Lemmas 6.2 and 6.3.

Theorem 6.2. Assume $P$ is formally symmetric and $I \subset D$ is a completely regular subspace for $P$. Define

$$X(u) = (Pu, u) - (2PU - AV, u) \quad . \tag{6.14}$$

Then $u \in D$ is a solution of the problem with linear restrictions (2.1), if and only if

$$(X'(u), v) = 0 \quad \forall v \in I \quad . \tag{6.15}$$



When $I$ is regular but not completely regular, an element $u \in V + I$ is a solution of (2.1), if and only if (6.15) holds.

Proof. $2P-A$ is symmetric with quadratic form $(2Pu, u)$, because $A$ is antisymmetric. From (6.14), it follows that

$$X'(u) = (2P-A)u - (2PU - AV) \quad . \tag{6.16}$$

Theorem 6.2, follows from Lemmas 6.1 and 6.3, by virtue of (6.16).

The following variational principles are corollaries of Theorems 6.1 and 6.2.

Theorem 6.3. Take $\hat{P}{:}\hat{D} \to \hat{D}^*$ as in Section 5 and let $\hat{S} \subset \hat{D}$ be a completely regular smoothness relation for $\hat{D}$. Define $\hat{J}{:}\hat{D} \to \hat{D}^*$ by means of (5.28). Then, when $\hat{P}$ is formally symmetric $\hat{u} \in \hat{D}$ is a solution of the problem of connecting (5.11), if and only if

$$\Omega'(\hat{u}) = 0 \tag{6.17}$$

where

$$\Omega(\hat{u}) = (\hat{P}(\hat{u} - 2\hat{U}), \hat{u}) - (\hat{J}(\hat{u} - 2\hat{V}), \hat{u}) \quad . \tag{6.18}$$

Proof. According to Theorem 5.1, the pair $(\hat{S}, \hat{M})$ constitutes a canonical decomposition of $\hat{D}$, where $\hat{M}$ is given by Definition 5.2. Hence, Theorem 6.3 follows Theorem 6.1, because $\hat{J}{:}\hat{D} \to \hat{D}^*$ is the operator that decomposes $\hat{A}$, associated by Theorem 4.1 with $(\hat{S}, \hat{M})$.

Theorem 6.4. Assume problem (2.1) is subjected to restrictions of continuation type (Definition 5.3) and the associated smoothness condition $\hat{S}$ is regular for $\hat{P}$. Let the functional $X{:}D \to F$ be given by (6.14). Then, when the problem of connecting satisfies existence and $P{:}D \quad D^*$ is formally symmetric, $u \in D$ fulfills (2.1), if and only if, (6.15) holds.

Proof. This result follows from Theorem 6.2, by virtue of Theorem 5.2

## 7. Applications.

The variational principles for the problem with linear constraints presented in Section 6, supply a systematic frame-work for the formulation of such principles associated with boundary value problems and boundary methods. There are many classical problems of partial differential equations that can be cast in this frame-work; here, however, it will only be applied to two types of problems: problems formulated in discontinuous fields subjected to prescribed jump conditions; and problems subjected to restrictions of continuation type. The corresponding variational principles will be special cases of Theorem 6.3 and 6.4, respectively.

These two kinds of principles will be derived for potential theory, reduced wave equation, heat and wave equations, and Elasticity (static, periodic motions and dynamical). Variational principles for the linearized theory of free surface flows have also been obtained by this method [Herrera, 1977a]. It is of interest to notice that problems involving two phases can also be formulated in this manner; to illustrate this fact variational principles are derived for a problem in which the region $R$ (Figure 1) is occupied by an inviscid liquid while $E$ is occupied by an elastic solid. For static and quasi-static problems the regions to be considered are illustrated in Figure 1. The regions illustrated in Figure 3 apply to time dependent problems, which will be formulated in a finite time interval $[0,T]$. For simplicity the regions $R$ and $E$ shown in the figures are bounded, but the results can also be applied in unbounded regions if suitable conditions such as radiation conditions are imposed on the elements of the spaces $D_R$ and $D_E$. Thus, for example, diffraction problems formulated in a half-space (Figure 4) can be treated in this manner.



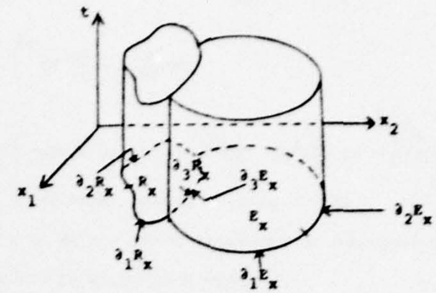


Figure 3


Figure 4

### 7.1. Potential theory and reduced wave equation.

The function spaces $D_R$ and $D_E$ can be taken as suitable Sobolev spaces [Babuska and Aziz, 1972; Lions and Magenes, 1968]; generally, $D_R = H^s(R)$, $D_E = H^s(E)$, with $s \geq 2$. A slight modification has to be made when complex valued functions are considered. Given $\rho$ and non-zero constants $k_R, k_E$, define

$$L(u) = \nabla^2 u + \rho u \tag{7.1}$$

$$(P_R u_R, v_R) = k_R \{ \int_R vL(u) d\underline{x} + \int_{\partial_1 R} u \frac{\partial v}{\partial n} d\underline{x} - \int_{\partial_2 R} v \frac{\partial u}{\partial n} d\underline{x} \} \tag{7.2}$$

and $P_E : D_E \to D_E^*$, replacing $R$ by $E$ in Equation 7.2. Then, using (5.10) it can be seen that

$$\hat{N} = \{(u_R, u_E) \in \hat{D} | u_R = u_E = \partial u_E/\partial n = \partial u_R/\partial n, \quad \text{on} \quad \partial_3 R\} \tag{7.3}$$

and it is easy to verify that $\hat{P}:\hat{D} \to \hat{D}^*$ is formally symmetric, because it satisfies (6.4).

Let the smoothness relation be

$$\hat{S} = \{\hat{u} \in \hat{D} | u_R = u_E, \; k_R \partial u_R/\partial n = k_E \partial u_E/\partial n; \quad \text{on} \quad \partial_3 R\} \quad . \tag{7.4}$$

For $\hat{v} \in \hat{S}$ and arbitrary $\hat{u} \in \hat{D}$

$$\langle \hat{A}\hat{u}, \hat{v} \rangle = \int_{\partial_3 R} \{ k[\hat{u}] \frac{\partial v}{\partial n} - v[\hat{k} \frac{\partial \hat{u}}{\partial n}]\} dx \tag{7.5}$$

where

$$[\hat{u}]_R = u_E - u_R \; ; \; [\hat{k} \; \partial \hat{u}/\partial n]_R = k_E \partial u_E/\partial n_R - k_R \partial u_R/\partial n_R \quad . \tag{7.6}$$

Here, as in what follows, the components (R or E) to be used when carrying out the integration, are indicated by the subindex under the integral sign. From (7.5) and Lemma 5.1, it can be seen that $\hat{S}$ is completely regular for $\hat{P}$. Applying (5.28), one gets

$$\langle \hat{J}\hat{u}, \hat{v} \rangle = \int_{\partial_3 R} \{\overline{k \frac{\partial v}{\partial n}} [\hat{u}] - \overline{v}[\hat{k} \frac{\partial \hat{u}}{\partial n}]\} dx \tag{7.7}$$

where

$$(\overline{v})_R = (u_E + u_R)/2 \; ; \; \left(\overline{k \frac{\partial v}{\partial n}}\right)_R = \frac{1}{2}\left(k_E \frac{\partial v_E}{\partial n_R} + k_R \frac{\partial v_R}{\partial n_R}\right) \quad . \tag{7.8}$$

Given $\hat{U} \in \hat{D}$ and $\hat{V} \in \hat{D}$, the problem of connecting (5.11), is equivalent to

$$L(\hat{u}) = f_{R \cup E} = L(\hat{U}) \quad ; \quad \text{on} \quad R \cup E \tag{7.9a}$$

$$\hat{u} = f_1 = \hat{U} \quad ; \quad \text{on} \quad \partial_1 (R \cup E) \tag{7.9b}$$

$$\frac{\partial \hat{u}}{\partial n} = f_2 = \frac{\partial \hat{U}}{\partial n} \quad ; \quad \text{on} \quad \partial_2 (R \cup E) \tag{7.9c}$$

subjected to prescribed jump conditions



$$u_E - u_R = f_{J1} = V_E - V_R \; ; \; k_E \frac{\partial u_E}{\partial n_R} - k_R \frac{\partial u_R}{\partial n_R} = f_{J2} = k_E \frac{\partial V_E}{\partial n_R} - k_R \frac{\partial V_R}{\partial n_R} \quad \text{on} \quad \partial_3 R \; . \tag{7.10}$$

This problem can be formulated variationally by means of Theorem 6.3. The corresponding functional is

$$\Omega(u) = \int_{R \cup E} u(u - 2f_{R \cup E}) dx + \int_{\partial_1 (R \cup E)} (u - 2f_1) \frac{\partial u}{\partial n} dx$$

$$- \int_{\partial_2 (R \cup E)} u(\frac{\partial u}{\partial n} - 2f_2) dx - \int_{\partial_3 R} \{([\hat{u}] - 2f_{J1}) \overline{k \frac{\partial u}{\partial n}} - \overline{u}([\hat{k} \frac{\partial \hat{u}}{\partial n} - 2f_{J2}) \} dx \quad . \tag{7.11}$$

The problem with restrictions of continuation type of Definition 5.3, in this case corresponds to

$$Lu = f_R = LU \quad ; \quad \text{on} \quad R \tag{7.12a}$$

$$u = f_1 = U \quad ; \quad \text{on} \quad \partial_1 R \tag{7.12b}$$

$$\frac{\partial u}{\partial n} - f_2 = \frac{\partial U}{\partial n} \quad ; \quad \text{on} \quad \partial_2 R \quad . \tag{7.12c}$$

The restriction is that there exists a function $u_E \in D_E$, such that

$$u - V = u_E \; ; \; k_R(\frac{\partial u}{\partial n} - \frac{\partial V}{\partial n}) = k_E \frac{\partial u_E}{\partial n}, \quad \text{on} \quad \partial_3 R \quad . \tag{7.13}$$

Here

$$Lu_E = 0, \quad \text{on} \quad E; \; u_E = 0, \quad \text{on} \quad \partial_1 E; \; \partial u_E/\partial n = 0, \quad \text{on} \quad \partial_2 E \quad . \tag{7.14}$$

This problem occurs in diffraction studies.

Taking $I \subset D$ as the linear subspace that satisfies (7.13); with $V \equiv 0$, Theorem 6.4 is applicable. Equation (6.14) yields

$$X(u) = \int_R u(Lu - 2f_R) dx + \int_{\partial_1 R} (u - 2f_1) \frac{\partial u}{\partial n} dx - \int_{\partial_2 R} u(\frac{\partial u}{\partial n} - 2f_2) dx$$

$$+ \int_{\partial_3 R} (u \frac{\partial V}{\partial n} - V \frac{\partial u}{\partial n}) dx \quad . \tag{7.15}$$

Here the factor $k_R$ was deleted because it was superfluous.

7.2. Heat equation.

A similar application can be made to the heat equation. In this case (Figure 3), $R = R_x \times [0,T]$, $E = E_x \times [0,T]$ and

$$\mathcal{L}u = \nabla^2 u - \partial u/\partial t \quad . \tag{7.16}$$

The operator $P_R:D_R \to D_R^*$ can be defined by

$$\langle P_R u_R, v_R\rangle = \int_{R_x} v * \mathcal{L}u dx + \int_{\partial_1 R_x} u * \frac{\partial v}{\partial n} dx - \int_{\partial_2 R_x} v * \frac{\partial u}{\partial n} dx$$
$$- \int_{R_x} u(0)v(T)dx \quad . \tag{7.17}$$

Here as in what follows the notation

$$u * v = \int_0^T u(T-t)v(t)dt \tag{7.18}$$

is adopted. $P_E:D_E \to D_E^*$ is obtained replacing $R$ by $E$ in (7.17). The smoothness condition can be taken as

$$\hat{S} = \{\hat{u} \in \hat{D} | u_R = u_E;\ \partial u_R/\partial n = \partial u_E/\partial n;\ \text{ on } \partial_3 R\} \tag{7.19}$$

where the subsets $\partial_i R = [0,T] \times \partial_i R_x$ $(i = 1,2,3)$, do not cover the boundary $\partial R$ of $R$. When $\hat{v} \in \hat{S}$ and $\hat{u} \in \hat{D}$ is arbitrary,

$$\langle \hat{A}\hat{u}, \hat{v}\rangle = \int_{\partial_3 R_x} \{[\hat{u}] * \frac{\partial v}{\partial n} - v * [\frac{\partial \hat{u}}{\partial n}]\}dx \quad . \tag{7.20}$$

Again, use of Lemma 6.1, permits establishing that $\hat{S}$ is completely regular for $\hat{D}:\hat{D} \to D^*$. Equation (5.28), yields

$$\langle \hat{J}\hat{u}, \hat{v}\rangle = \int_{\partial_3 R_x} \{ \overline{\frac{\partial v}{\partial n}} * [\hat{u}] - \overline{v} * [\frac{\partial \hat{u}}{\partial n}]\}dx \quad . \tag{7.21}$$



Given $\hat{U} \in \hat{D}$ and $\hat{V} \in \hat{D}$, the problem of connecting (5.11), is equivalent to Equation (7.9) supplemented by

$$\hat{u}(x,0) = f_0 = \hat{U}(x,0);\ \text{ on } R_x \cup E_x \tag{7.22}$$

subjected to jump conditions

$$u_E - u_R = f_{J1} = V_E - V_R,\ \frac{\partial u_E}{\partial n} - \frac{\partial u_R}{\partial n} = f_{J2} = \frac{\partial V_E}{\partial n} - \frac{\partial V_R}{\partial n};\ \text{ on } \partial_3 R \quad . \tag{7.23}$$

Here, $\partial_3 R = [0,T] \times \partial_3 R_x$; thus, $f_{J1}$ and $f_{J2}$ are functions of time $t$, also.

The variational formulation of Theorem 6.3, yields the functional

$$\Omega(\hat{u}) = \int_{R_x \cup E_x} u * (\mathcal{L}u - 2f_{R\cup E})dx + \int_{\partial_1(R_x \cup E_x)} (u - 2f_1) * \frac{\partial u}{\partial n} dx - \int_{\partial_2(R_x \cup E_x)} u * (\frac{\partial u}{\partial n} - 2f_2)dx$$
$$- \int_{R_x \cup E_x} \{u(0) - 2f_0\}u(T)dx + \int_{\partial_3 R_x} (\overline{u} * ([\frac{\partial \hat{u}}{\partial n}] - 2f_{J2}) - ([\hat{u}] - 2f_{J1}) * \overline{\frac{\partial u}{\partial n}} dx \quad . \tag{7.24}$$

The problem with restrictions of continuation type of Definition 5.3, in this case is governed by equations (7.12), supplemented by

$$u(x,0) = f_0 = U(x,0);\ \text{ on } R_x \quad . \tag{7.25}$$

The restriction is obtained taking $k_R = k_E = 1$ in Equation (7.13) and supplementing (7.14) with

$$u_E(x,0) = 0 \ ;\ \text{ on } E_x \quad . \tag{7.26}$$

The functional of Theorem 6.4, is

$$X(u) = \int_{R_x} u * (\mathcal{L}u - 2f_R)dx + \int_{\partial_1 R_x} (u - 2f_1) * \frac{\partial u}{\partial n} dx - \int_{\partial_2 R} u * (\frac{\partial u}{\partial n} - 2f_2)dx$$
$$- \int_{R_x} \{u(0) - 2f_0\}u(T)dx + \int_{\partial_3 R_x} \{u * \frac{\partial V}{\partial n} - V * \frac{\partial u}{\partial n}\}dx \quad . \tag{7.27}$$

7.3. The Wave Equation.

The results corresponding to the wave equation are listed below.

a). $R = R_x \times [0,T]$; $E = E_x \times [0,T]$

b). $\mathcal{L}u = \nabla^2 u - \partial^2 u/\partial t^2$ (7.28)

c). $P_R : D_R \to D_R^*$ is

$$(P_R u_R, v_R) = \int_{R_x} v\, \mathcal{L}u\,d\underset{\sim}{x} + \int_{\partial_1 R_x} u \cdot \frac{\partial v}{\partial n}\,d\underset{\sim}{x} - \int_{\partial_2 R_x} v \cdot \frac{\partial u}{\partial n}\,d\underset{\sim}{x} \qquad (7.29)$$

$$- \int_{R_x} \{u(0)v'(T) + u'(0)v(T)\}d\underset{\sim}{x}$$

where the primes stand for the partial derivatives with respect to $t$. To obtain $P_E : D_E \to D_E^*$, $R$ has to be replaced by $E$ in (7.29)

d). Equations (7.19) to (7.21) also hold in this case.

e). Given $\hat{U} \in \hat{D}$ and $\hat{V} \in \hat{D}$, the problem of connecting (5.11), is equivalent to Equation (7.9), supplemented by

$$\hat{u}(\underset{\sim}{x},0) = f_0 = \hat{U}(\underset{\sim}{x},0);\ \partial\hat{u}(\underset{\sim}{x},0)/\partial t = f_0' = \partial\hat{U}(\underset{\sim}{x},0)/\partial t; \quad \text{on } R_x \cup E_x \qquad (7.30)$$

subjected to (7.23).

f). The functional of Theorem 6.3, is

$$\Sigma(\hat{u}) = \int_{R_x \cup E_x} u \cdot (\mathcal{L}u - 2f_{R \cup E})d\underset{\sim}{x} + \int_{\partial_1(R_x \cup E_x)} (u - 2f_1) \cdot \frac{\partial u}{\partial n}\,d\underset{\sim}{x} - \int_{\partial_2(R_x \cup E_x)} u \cdot \left(\frac{\partial u}{\partial n} - 2f_2\right)d\underset{\sim}{x}$$

$$- \int_{R_x \cup E_x} \{u(0) - 2f_0\}u'(T)d\underset{\sim}{x} - \int_{R_x \cup E_x} \{u'(0) - 2f_0'\}u(T)d\underset{\sim}{x} \qquad (7.31)$$

$$+ \int_{\partial_3 R_x} \{\bar{u} \cdot ([\tfrac{\partial \hat{u}}{\partial n}] - 2f_{J2}) - ([\hat{u}] - 2f_{J1}) \cdot \overline{\tfrac{\partial u}{\partial n}}\}d\underset{\sim}{x} \quad .$$

g). The problem with restrictions of continuation type of Definition 5.3, is given by (7.12), (7.25), supplemented by



$$\partial u(\underset{\sim}{x},0)/\partial t = f_0' = \partial U(\underset{\sim}{x},0)/\partial t; \quad \text{on } R_x \qquad (7.32)$$

subjected to the restriction that there exists $u_E \in D_E$, that satisfies (7.13) with $k_R = k_E = 1$, (7.14) and (7.26), together with

$$\partial u_E(\underset{\sim}{x},0)/\partial t = 0 \quad ; \quad \text{on } E_x \quad . \qquad (7.33)$$

h.) The functional of Theorem 6.4, is

$$X(u) = \int_{R_x} u \cdot (\mathcal{L}u - 2f_R)d\underset{\sim}{x} + \int_{\partial_1 R_x} (u - 2f_1) \cdot \frac{\partial u}{\partial n}\,d\underset{\sim}{x} - \int_{\partial_2 R} u \cdot \left(\frac{\partial u}{\partial n} - 2f_2\right)d\underset{\sim}{x} \qquad (7.34)$$

$$- \int_{R_x} \{u(0) - 2f_0\}u'(T)d\underset{\sim}{x} - \int_{R_x} \{u'(0) - 2f_0'\}u(T)d\underset{\sim}{x} + \int_{\partial_3 R_x} \left\{u \cdot \frac{\partial V}{\partial n} - V \cdot \frac{\partial u}{\partial n}\right\}d\underset{\sim}{x} \quad .$$

7.4. Elasticity.

In order to formulate the problems of Elasticity, the elastic tensor $C_{ijpq}$ is assumed to be defined in the regions $R$ and $E$. It will be assumed to be sufficiently differentiable on $R$ and on $E$, separately; for example, *it is not too restrictive to assume that* $C_{ijpq}$ possesses continuous derivatives of all orders on $R$ and on $E$, that can be extended continuously to the boundaries of these regions. *In addition,* $C_{ijpq}$ is assumed to satisfy the usual symmetry conditions [Gurtin, 1972]

$$C_{ijpq} = C_{pqij} = C_{jipq} \qquad (7.35)$$

and to be strongly elliptic; i.e.

$$C_{ijpq}\xi_i\eta_j\xi_p\eta_q > 0, \quad \text{whenever} \quad \xi_i\xi_i \neq 0;\ \eta_i\eta_i \neq 0 \quad . \qquad (7.36)$$

7.4.1. Static and periodic motions.

The elements of the linear spaces $D_R$ and $D_E$, can be taken as vector valued functions whose components belong to $H^s(R)$ and $H^s(E)$, $s \geq 2$,

respectively. When treating periodic motions in unbounded domains, it is frequently convenient to consider vector valued vector fields. Let

$$\tau_{ij}(\underset{\sim}{u}) = C_{ijpq}\frac{\partial u_p}{\partial x_q}; \quad \text{on} \quad R \cup E \tag{7.37a}$$

$$L_i(\underset{\sim}{u}) = \frac{\partial \tau_{ij}}{\partial x_j} + ku_i; \quad \text{on} \quad R \cup E \tag{7.37b}$$

and

$$T_i(\underset{\sim}{u}) = \tau_{ij}(\underset{\sim}{u})n_j \quad ; \quad \text{on} \quad \partial R \quad \text{and} \quad \partial E \quad . \tag{7.37c}$$

Here $k$ is a function of position which satisfies continuity conditions similar to those of the elastic tensor. The definition of the tractions $\underset{\sim}{T}(\underset{\sim}{u})$ depends on the sense of the unit normal vector, so that two such tractions which have opposite signs, are defined on the common boundary $\partial R = \partial E$. As in the case of the normal vector, sometimes they will be represented by $\underset{\sim}{T}_R(\underset{\sim}{u})$ and $\underset{\sim}{T}_E(\underset{\sim}{u})$; more often, however, the ambiguity will be resolved by the suffix used under the integral sign. Observe that when considering the problem of connecting the following combinations can occur $\underset{\sim}{T}_R(\underset{\sim}{u}_R)$, $\underset{\sim}{T}_R(\underset{\sim}{u}_E)$, $\underset{\sim}{T}_E(\underset{\sim}{u}_R)$ and $\underset{\sim}{T}_E(\underset{\sim}{u}_E)$.

The definitions and results for static and periodics motions in Elasticity are listed below:

a.) $P_R : D_R \to D_R^*$ is

$$\langle P_R u_R, v_R \rangle = \int_R v_i L_i(\underset{\sim}{u})d\underset{\sim}{x} + \int_{\partial_1 R} u_i T_i(\underset{\sim}{v})d\underset{\sim}{x} - \int_{\partial_2 R} v_i T_i(\underset{\sim}{u})d\underset{\sim}{x} \tag{7.38}$$

and $P_E : D_E \to D_E^*$ is obtained replacing $R$ by $E$ in (7.38).

b.) The smoothness condition can be taken as

$$\hat{S} = \{\hat{u} \in \hat{D} \,|\, u_{Ri} = u_{Ei};\ T_i(\underset{\sim}{u}_R) = T_i(\underset{\sim}{u}_E); \quad \text{on} \quad \partial_3 R\} \quad . \tag{7.39}$$



c.) When $\hat{v} \in \hat{S}$ and $\hat{u} \in \hat{D}$ is arbitrary

$$\langle \hat{A}\hat{u}, \hat{v} \rangle = \int_{\partial_3 R} \{[\hat{u}_i]T_i(\underset{\sim}{v}) - v_i[T_i(\hat{u})]\}d\underset{\sim}{x} \quad . \tag{7.40}$$

Here

$$[\hat{u}_i] = u_{Ei} - u_{Ri} \ ; \ [T_i(\hat{u})]_R = T_{Ri}^E(\underset{\sim}{u}_E) - T_{Ri}^R(\underset{\sim}{u}_R) \quad . \tag{7.41}$$

Where the subindices $R$ and $E$ in the tractions, refer to the normal vector used, while the superindices refer to the elastic tensor used; thus, for example $T_{Ri}^E(\underset{\sim}{u}_E) = C_{ijpq}^E \frac{\partial u_{Ep}}{\partial x_q} n_{Rj}$ .

d.) $\hat{S}$ is completely regular for $\hat{P} : \hat{D} \to \hat{D}^*$. This result can be established using Lemma 5.1 and strong ellipticity (Equation 7.36).

e.) Equation (5.28), yields

$$\langle \hat{J}\hat{u}, \hat{v} \rangle = \int_{\partial_3 R} \{[\hat{u}_i]\overline{T_i(\underset{\sim}{v})} - \bar{v}_i[T_i(\underset{\sim}{u})]\}d\underset{\sim}{x} \quad . \tag{7.42}$$

f.) Given any $\hat{U} \in \hat{D}$ and $\hat{V} \in \hat{D}$, the problem of connecting (5.11) is equivalent to

$$L_i\hat{u} = f_{R\cup Ei} = L_i\hat{U} \quad ; \quad \text{on} \quad R \cup E \tag{7.43a}$$

$$\hat{u}_i = f_{1i} = \hat{U}_i \quad ; \quad \text{on} \quad \partial_1(R \cup E) \tag{7.43b}$$

$$T_i(\hat{u}) = f_{2i} = T_i(\hat{U}); \quad \text{on} \quad \partial_2(R \cup E) \tag{7.43c}$$

subjected to the jump conditions

$$[\hat{u}] = \underset{\sim}{f}_{J1} = [\hat{V}], \ [\underset{\sim}{T}(\hat{u})] = \underset{\sim}{f}_{J2} = [\underset{\sim}{T}(\underset{\sim}{V})]; \quad \text{on} \quad \partial_3 R \quad . \tag{7.44}$$

g.) For this problem, the variational formulation of Theorem 6.3 yields the functional

$$\Omega(\hat{u}) = \int_{R\cup E} u_i(L_i u - 2f_{R\cup Ei})dx + \int_{\partial_1 R\cup E)} (u_i - 2f_{1i})T_i u\, dx$$

$$- \int_{\partial_2(R\cup E)} u_i(T_i u - 2f_{2i})dx + \int_{\partial_3 R} \{\bar{u}_i([T_i(u)] - 2f_{J2i}) \qquad (7.45)$$

$$- ([u_i] - 2f_{J1i})\overline{T_i(u)}\}dx \;.$$

h.) The functional of Theorem 6.4, for the problem with restrictions of continuation type is given by

$$X(u) = \int_R u_i(L_i u - 2f_{Ri})dx + \int_{\partial_1 R} (u_i - 2f_{1i})T_i u\, dx$$

$$- \int_{\partial_2(R\cup E)} u_i(T_i u - 2f_{2i})dx + \int_{\partial_3 R} \{u_i T_i(v) - V_i T_i(u)\}dx \;. \qquad (7.46)$$

7.4.2. Dynamics.

The extension from elastostatics to dynamic elasticity is very similar to *that carried* out when going to the wave equation from Laplace's. The operators have to be defined as

$$D_i u = L_i u - \rho \frac{\partial^2 u_i}{\partial t^2} \qquad (7.47)$$

where $L_i$ is given by (7.37b) with $k \equiv 0$; then

$$(P_R u_R, v_R) = \int_{R_x} v_i * D_i u\, dx + \int_{\partial_1 R_x} u_i * T_i(v)dx - \int_{\partial_2 R_x} v_i * T_i(u)dx$$

$$- \int_{R_x} \rho\{u_i(0)v_i'(T) + u_i'(0)v_i(T)\}dx \qquad (7.48)$$

where, as in (7.29), the primes stand for the partial derivatives with respect to time. The regions are shown in Figure 3. The smoothness condition is given (7.39), with the new interpretation of $\partial_3 R$. It can be shown that $\hat{S}$ is completely regular for $\hat{P}:\hat{D} \to \hat{D}^*$, so that Theorems 6.3 and 6.4 can be applied.

For the jump operator, it is obtained

$$\langle \hat{J}u, \hat{v}\rangle = \int_{\partial_3 R} \{[u_i]*\overline{T_i(v)} - \bar{v}_i*[T_i(u)]\}dx \;. \qquad (7.49)$$



7.5. A two phase problem.

Let $R$ in Figure 1, be occupied by a linear elastic solid, while $E$ will be occupied by an inviscid compressible fluid. It will be assumed that the motion in $E$ is potential and the governing equations have been linearized.

For periodic motions of angular frequency $\omega$, *Equation (7.43) apply on* $R$, with $k = \rho\omega^2$. In general, when the motion is non-periodic, the equations in $R$ are [Meyer, 1972; Landau and Lifshitz, 1959]

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0 \;;\; \text{on } R \qquad (7.47a)$$

*where* $p$ *is the pressure and* $c^2 = (dp/d\rho)_0$ will be taken as constant. The acceleration $a_i$ is

$$a_i = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} \;. \qquad (7.47b)$$

The smoothness conditions across the connecting boundary $\partial_3 R = \partial_3 E$ are: continuity of fractions and continuity of normal components of displacements. For periodic motions $p = u_E e^{i\omega t}$, this leads to

$$u_E n_i + T_i(u) = 0 \;;\; \text{on } \partial_3 R \qquad (7.48a)$$

$$\frac{\partial u_E}{\partial n} - \rho\omega^2 u_i n_i = 0; \;\text{on } \partial_3 R \;. \qquad (7.48b)$$

The inhomogeneous form of (7.47a), for such periodic motions, is

$$\nabla^2 u_E + \rho\omega^2 u_E = f_E; \;\text{on } E \;. \qquad (7.49)$$

Therefore, the problem is governed by (7.43), on $R$ and (7.49), subjected to the smoothness conditions (7.48). In order to consider the more general problem, for which the right-hand side in Equation (7.48) may be prescribed non-zero functions, the operator $P_R:D_R \to D_R^*$ will be defined multiplying the

right-hand side of (7.38) by $\rho\omega^2$, while $P_E : D_E \to D_E^*$ is defined replacing $R$ by $E$ in (7.2). Notice that functions of $D_R$ are vector valued, while those of $D_E$ have only one component. Then

$$\langle \hat{\hat{A}}\hat{u}, \hat{v}\rangle = \rho\omega^2 \int_{\partial_3 R} \{v_i T_i(\underline{u}) - u_i T_i(\underline{v})\} d\underline{x} + \int_{\partial_3 E} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n}\right) d\underline{x} \quad . \qquad (7.50)$$

The smoothness relation $\hat{\mathcal{S}} \subset \hat{D}$ is defined as the set whose elements satisfy (7.48). When $\hat{v} = (v_R, v_E) \in \hat{\mathcal{S}}$, while $\hat{u} = (u_R, u_E) \in \hat{D}$ is arbitrary, Equation (7.50) reduces to

$$\langle \hat{\hat{A}}\hat{u}, \hat{v}\rangle = \rho\omega^2 \int_{\partial_3 R} v_i \{T_i(\underline{u}) + u_E n_i\} d\underline{x} - \int_{\partial_3 R} T_i(\underline{v}) \left\{\frac{\partial u_E}{\partial n} n_i - \rho\omega^2 u_i\right\} d\underline{x} \quad . \qquad (7.51)$$

When strong ellipticity (7.36) is satisfied, it can be shown that $v_i$ and $T_i(\underline{v})$ can be varied independently. Using this fact and Equation (7.51), it is not difficult to see that

$$\langle \hat{\hat{A}}\hat{u}, \hat{v}\rangle = 0 \ \forall \hat{v} \in \hat{\mathcal{S}} \Rightarrow \hat{u} = (u_R, u_E) \text{ satisfies (7.48)} \ ; \qquad (7.52)$$

hence, $\hat{\mathcal{S}}$ is completely regular for $\hat{P}$. That $\hat{P}$ is formally symmetric follows from the fact that

$$\hat{N} = \{\hat{u} \in \hat{D} \,|\, u_{Ri} = u_E = \partial u_{Ri}/\partial n = \partial u_E/\partial n = 0; \text{ on } \partial_3 R \qquad (7.53)$$

which involves boundary conditions on $\partial_3 R$, only. Thus, the general theory developed previously is applicable to $\hat{P}$ and the variational principles of Theorem 6.3 and 6.4 are applicable to this problem. It is now a straight-forward exercise to obtain corresponding formulas, but the details will not be included here.



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A general theory of problems subjected to linear constraints is developed. As applications, a problem for which the solutions are required to satisfy prescribed jumps and another one whose solutions are restricted to be such that they can be continued smoothly into solutions of given equations in neighboring regions, are formulated abstractly. General variational principles for these types of problem are reported. In addition it is shown that sets of functions that can be extended in the manner explained above constitute, generally, completely regular subspaces, here defined. These results have a bearing on




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ABSTRACT (continued)

boundary methods which are being developed for treating numerically partial differential equations associated with many problems of Science and Engineering.